INEQUALITIES

UNIT 3 GEOMETRIC INEQUALITIES

A **geometric inequality** is an inequality which appears in a geometric context. The simplest examples are those involving variables which are sides of triangles that we came across in Unit 2. In this unit we shall focus on geometric inequalities.

As before we will first study some classical geometric inequalities. After that we will illustrate the concepts by examples.

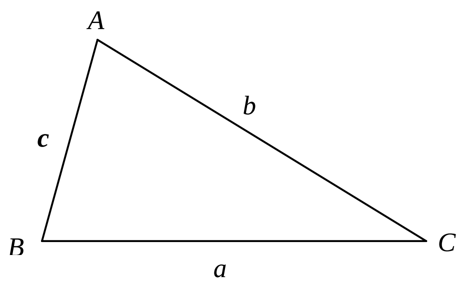
1. Basic Concepts in Geometry

Although this is not a topic in geometry, some basic knowledge in geometry is essential in order for us to study geometric inequalities.

As usual, given ΔABC, we denote by a, b, c the lengths of the sides opposite A, B and C. We also denote

$$s = \frac{1}{2}(a+b+c)$$

to be the semi-perimeter of ΔABC. R and r denote the circumradius and in-radius of ΔABC, respectively. We denote by $[ABC]$ the area of ΔABC.



The following are some basic theorems in geometry.

Theorem 1.1. (Sine Law and Cosine Law)

With the above notations,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Theorem 1.2. (Area of Triangle)

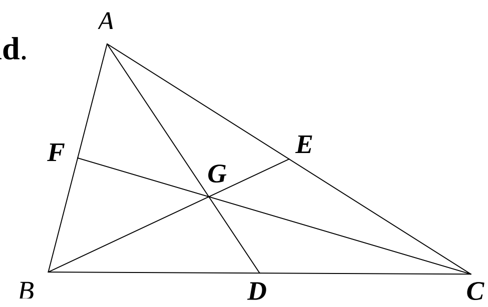
The area of ΔABC can be expressed in the following ways.

$$[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$$
$$[ABC] = \frac{abc}{4R}$$
$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{(Heron's Fornula)}$$
$$[ABC] = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = sr$$

Theorem 1.3. (Centroid Theorem)

The three **medians** of a triangle are concurrent at the **centroid**. Moreover, the centroid divides the median internally in the ratio $2:1$, i.e. if *D*, *E*, *F* are the mid-point of *BC*, *CA* and *AB* respectively, then *AD*, *BE* and *CF* are concurrent at some point *G*. Moreover,

$$\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = 2.$$



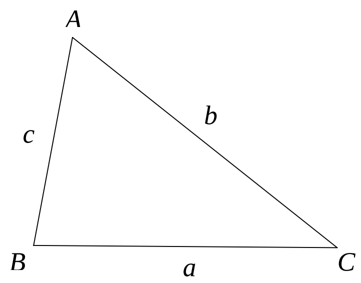
2. Classical Geometric Inequalities

Many geometric inequalities can be interpreted in an algebraic context and proved by one of the ‘Big Three’— the AM-GM inequality, the Cauchy Schwarz inequality and the rearrangement inequality. However, it is useful to learn a few more inequalities which are more ‘geometrical’ in nature.

Theorem 2.1. (Triangle Inequality)

Given ΔABC with side lengths *a*, *b*, *c*,

$$a + b > c$$
$$b + c > a$$
$$c + a > b$$



Theorem 2.2. (Euler)

Let R and r denote the circumradius and in-radius respectively of the same triangle. Then

$$R \geq 2r\,.$$

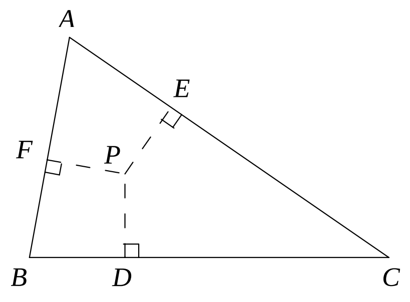
Equality holds if and only if the triangle is equilateral.

Theorem 2.3. (Erdös-Mordell)

Let P be a point inside ΔABC; D, E, F be the feet of the perpendiculars from P to BC, CA and AB respectively. Then

$$PA + PB + PC \geq 2(PD + PE + PF)\,.$$

Equality holds if and only if ΔABC is equilateral.

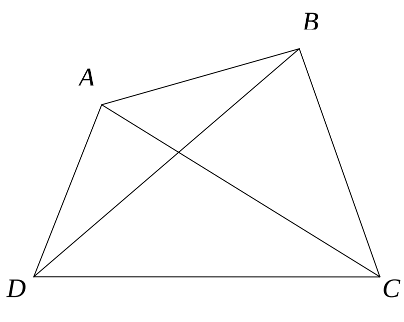


Theorem 2.4. (Ptolemy)

Let $ABCD$ be a quadrilateral. Then

$$AB \times CD + AD \times BC \geq AC \times BD\,.$$

Equality holds if and only if $ABCD$ is cyclic.



3. Worked Examples

Example 3.1.

A wire of length *p* is bent to form a triangle. What is the maximum possible area of the triangle?

Solution.

As usual, let $p = 2s$. By the AM-GM inequality,

$$\begin{aligned}[ABC] &= \sqrt{s(s-a)(s-b)(s-c)} \\ &\le \sqrt{s\left[\frac{(s-a)+(s-b)+(s-c)}{3}\right]^3} \\ &= \frac{\sqrt{3}}{9}s^2\end{aligned}$$

Equality holds if and only if $s-a = s-b = s-c$, i.e. $a = b = c$.

Hence the maximum area is $\frac{\sqrt{3}}{9}s^2$ or $\frac{\sqrt{3}}{36}p^2$, when the triangle is equilateral.

Example 3.2.

(IMO 1961) Let *a*, *b*, *c* be the sides of a triangle, and *T* its area. Prove

$$a^2 + b^2 + c^2 \ge 4\sqrt{3}T.$$

In what case does equality hold?

Solution.

By the previous example and the Cauchy-Schwarz inequality,

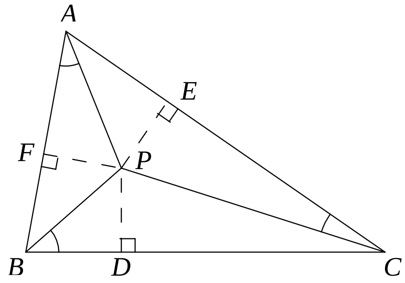
$$\begin{aligned}4\sqrt{3}T &\le 4\sqrt{3}\left[\frac{\sqrt{3}}{4}\left(\frac{a+b+c}{3}\right)^2\right] \\ &= \frac{1}{3}(a+b+c)^2 \\ &\le \frac{1}{3}(a^2+b^2+c^2)(1^2+1^2+1^2) \\ &= a^2+b^2+c^2\end{aligned}$$

In both applications of inequalities, equality holds if and only if the triangle is equilateral.

Example 3.3.

(IMO 1991) Let *ABC* be a triangle and *P* an interior point of ΔABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to 30°.

Solution.



Let *D*, *E*, *F* be the feet of the perpendicular from *P* to *BC*, *CA*, *AB* respectively. Suppose $\angle PAB$, $\angle PBC$ and $\angle PCA$ are all greater than 30°. Then we have

$$\begin{aligned} PA+PB+PC &= \frac{PF}{\sin\angle PAB}+\frac{PD}{\sin\angle PBC}+\frac{PE}{\sin\angle PCA} \\ &< \frac{PF}{\sin 30^\circ}+\frac{PD}{\sin 30^\circ}+\frac{PE}{\sin 30^\circ} \\ &= 2(PD+PE+PF) \end{aligned}$$

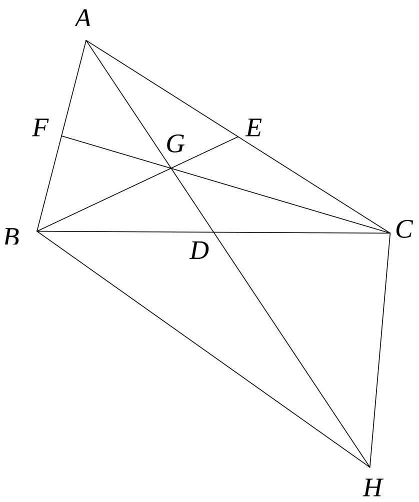
contradicting the Erdös-Mordell inequality. The result follows.

Example 3.4.

Let *ABC* be a triangle, *a*, *b*, *c* denote the lengths of the sides opposite *A*, *B*, *C* respectively; m_a, m_b, m_c denote the lengths of the medians from *A*, *B*, *C* respectively. Show that

$$\frac{3}{4}(a+b+c) < m_a+m_b+m_c < a+b+c\,.$$

Solution.



Let *D*, *E*, *F* be the mid-points of *BC*, *CA*, *AB* respectively, and *G* be the centroid of ΔABC.

In ΔABG, $GA+GB>AB$, i.e.

$$\frac{2}{3}m_a+\frac{2}{3}m_b>c\,.$$

Similarly,

$$\frac{2}{3}m_b+\frac{2}{3}m_c>a$$

$$\frac{2}{3}m_c+\frac{2}{3}m_a>b$$

Adding these inequalities, we have

$$\frac{4}{3}\left(m_a+m_b+m_c\right)>a+b+c\,,$$

i.e.

$$m_a + m_b + m_c > \frac{3}{4}(a+b+c).$$

This proves the left-hand inequality.

For the right-hand inequality, produce AD to H such that $AD = DH$.

In ΔABH , $BA + BH > AH$. Noting that $ABHC$ is a parallelogram, this is equivalent to

$$c + b > 2m_a .$$

Similarly, we have

$$a + c > 2m_b$$
$$b + c > 2m_a$$

Adding and dividing by 2, we get

$$a + b + c > m_a + m_b + m_c$$

as desired.

4. Exercises

1. Many different solutions exist for Example 3.2. Try to figure out another solution.

2. Let a, b, c be positive real numbers such that $a^2 + b^2 - ab = c^2$. Prove that

$$(a-c)(b-c) \le 0 .$$

3. (IMO 1981) P is a point inside a given triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.

4. (IMO 1991) Given a triangle ABC, let I be the centre of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides at A', B', C', respectively. Prove that

$$\frac{1}{4} \le \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \le \frac{8}{27}.$$

5. (IMO 1995) Let *ABCDEF* be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$ and $\angle BCD = \angle EFA = 60°$. Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°$. Prove that

$$AG + GB + GH + DH + HE \ge CF.$$

6. (IMO 1996) Let *ABCDEF* be a convex hexagon such that *AB* is parallel to *ED*, *BC* is parallel to *FE* and *CD* is parallel to *AF*. Let R_A, R_C, R_E denote the circumradii of triangles *FAB*, *BCD*, *DEF* respectively, and let p denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \ge \frac{p}{2}.$$